Indian J Phys. 75B (5), 413–418 (2001)

I J P B
an international journal

Sine-Gordon solitons in a nontrivial metric

K Javidan* and M Sarbishaei
Department of Physics, Ferdowsi University of Mashhad, Iran
E-mail javidan@science1.um.ac.ir

Received 30 November 2000, accepted 29 June 2001

Abstract We study the behavior of soliton solutions of the sine-Gordon equation with a nontrivial metric We can study the impurity interactions of a soliton by looking at the impurity as a nontrivial medium in which the soliton propagatges Therefore, we add a potential-like term $v(x)$ to the metric Our numerical simulations show that with the presence of a small potential-like $v(x)$ in the metric, the conserved quantities of system of solitons, like topological charge and total energy remain unchanged We study the evolution of soliton in some cases where $v(x)$ is a well or a barrier We find that the soliton behaves like a particle in most of the cases

Keywords Sine-Gordon solitons, nontrivial metric, numerical simulations

PACS Nos. 05.45 Yv, 03 10 -a

1. Introduction

Relativistic solitons, including those of the conventional sine-Gordon equation exhibit remarkable similarities with classical point-like particles. They exert forces on each other and make collisions, without losing their identities [1] Yet they are waves with infinite degrees of freedom which are localized and do not disperse while propagating in the medium For these reasons, they become so important in the description of phenomena like optical self-focusing, magnetic flux in Josephson junction [2] or even the very existence of stable elementary particles, such as the skyrmion [3,4] as a model of hadrons. The behavior of solitons in an interaction with impurities of medium is very important and also interesting both in theoretical and in applied physics. We can add the effects of disorders or impurities of medium as a perturbative term to the equation of motion. We can also take these effects by getting some parameters of the equation as a function of space or time [5,6]. In addition to these methods we can visualize these interactions by introducing a nontrivial metric for the relevant spacetime. Thus, the metric carries the information of the medium characters. We add a potential-like term $v(x)$ to the metric. It is also a good way to couple the soliton to an external potential without spoiling the topological boundary conditions. In this paper, we use this method for studying the interaction of the sine-Gordon soliton with impurity of medium using a suitable metric. In Section 2, we review the essential equations in a space-dependent metrics. In Section 3, we use the results of Section 2 for studying the evolution of sine-Gordon solitons in (1+1) dimensions with some specific $v(x)$, such as well, single barrier and double barrier. We use numerical simulation for solving the equations of motion and evaluate the topological charge and energy. We observe that if a soliton with an initial velocity greater than a certain value, impinges onto a potential barrier, it can pass the barrier; otherwise, it will be reflected If soliton is placed in a well, depending on its initial velocity and specification of the well, it may oscillate in the well or leaves the well. Our numerical simulation, also show bound states for two-kink solution of the sine-Gordon equation in a double barrier potential. Finally, we study the evolution of sine-Gordon solitons in presence of Delta-like potential. This case exhibits interesting effects, which is not found for particles.

2. Lagrange's equation in a space dependent metric

The action integral in a curved space with metric of $g^{\mu\nu}(x)$ is [7]

$$I = \int L(\varphi, \partial_\mu \varphi)\, dx dt \,, \tag{1}$$

*Corresponding Author

where $l(\varphi, \partial_\mu \varphi)$ is the Lagrangian density of the sytem and g is determinant of $g^{\mu\nu}(x)$ We can extract the Lagrange's equation from (1) as the following

$$I = \int l(\varphi, \partial_\mu \varphi)\sqrt{-g}\, dx\, dt \tag{2}$$

For Lagrangian of

$$l = \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - u(\varphi), \tag{3}$$

the equation of motion is

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\partial_\mu \partial^\mu \varphi + \partial_\mu \varphi \partial^\mu \left(\sqrt{-g}\right)\right) + \frac{\partial u}{\partial \varphi} = 0. \tag{4}$$

Now, we consider the metric of spacetime as

$$g_{00} = 1 + v(x), \quad g_{01} = g_{10} = 0, \quad g_{11} = -1 \tag{5}$$

or,
$$g_{\mu\nu}(x) = \begin{pmatrix} 1 + v(x) & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

In fact for a background weak potential independent of time, the metric with some approximation is [8 10]

$$g_{00} \approx 1 + v(x), \quad g_{01} = g_{10} = 0, \quad g_{11} = -1. \tag{7}$$

Using (6), we can find the Hamiltonian density and total energy as

$$\chi = \sqrt{-g}\left(\frac{1}{2} g_{00}\dot{\phi}^2 + \frac{1}{2}\varphi'^2 + u(\varphi)\right) \tag{8}$$

$$E = \int \chi \sqrt{+1}\, dx = \int \chi\, dx. \tag{9}$$

The usual Lagrangian density for sine-Gordon equation

$$l = \frac{m^4}{\lambda}\left[\frac{1}{2}(g_{\mu\nu}\partial^\mu \varphi \partial^\nu \varphi) + (\cos\varphi - 1)\right] \tag{10}$$

For $m = 1$ and $\lambda = 1$, the equation of motion for this Lagrangian density is as follows :

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + \sin\varphi = 0. \tag{11}$$

The static localized solutions for this equation are

$$\varphi(x) = 4\tan^{-1}\left[\pm\exp(x - x_0)\right]. \tag{12}$$

The solution with plus (minus) sign goes from $\varphi = 0(\varphi = 2\pi)$ to $\varphi = 2\pi(\varphi = 0)$ or equivalently from $2\pi(4\pi)$ to $4\pi(2\pi)$ *etc* Moving soliton solutions can be obtained on Lorentz-transforming (12), *i e.*, on replacing $(x - x_0)$ by $[(x - x_0 - ut)/\sqrt{1-u^2}]$. If we use (2) and (4) with metric (6) for the Lagrangian (10), the equation of motion becomes

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{2|1 + v(x)|}\frac{\partial \varphi}{\partial x}\frac{\partial v(x)}{\partial x} + \sin\varphi = 0. \tag{13}$$

If v(x) (which we call the potential) is weak enough, we can use the usual soliton solutions of the sine-Gordon equation in $t = 0$ for the initial condition and then solve the equation of motion numerically. For the sine-Gordon equation, the topological charge in our metric, becomes [11]

$$Q = \int \frac{\partial \varphi}{\partial x} dx. \tag{14}$$

3. Numerical techniques and results

This model generally is a non-integrable system so explicit solutions to its resulting differential equations are almost impossible to find. Numerical methods are the best way to solve this equation. We need eq. (13) for time evolution of an initial configuration. We use the finite difference method for solving the equation of motion and calculating the charge and energy of system. We take the time steps to be half or less than lattice spacing : $\delta t \leq \frac{1}{2}\delta x$ We use the solution (12) as our initial condition and keep the values of field in the boundaries of lattice fixed (fix boundary condition) To get a reasonable result, we use a lattice with 1000 points (or higher) with $\delta x = 0.01$ We check our results in every simulation *via* quantities conserved in the continum limit and by changing lattice spacing and number of points. We also solve the eq. (13) by using 4th order Runge-Kutta method as an alternative technique It is important that interaction of soliton takes place far from the boundaries. In this section we will find the time evolution of soliton solutions of the sine-Gordon equation and its topological charge and energy with different $v(x)$ potentials

3.1 *Conservation of topological charge and energy in presence of a weak potential .*

We have compared the evolution of topological charge in absence of potential ($v(x) = 0$) with topological charge in the presence of $v(x)$ potentials We use $v(x) = ae^{-bx^2}$ for simulating barrier, well and Delta function potentials. For small positive values of b (between 1 and 5), if $a < 0$, then $v(x)$ simulates a potential well; for $a > 0$, $v(x)$ behaves like a barrier. If a and b are large and positive, the potential simulates a Delta function Figure 1 shows the topological charge as a function of time for a kink solution with zero initial velocity, for $v(x) = ae^{-5x^2}$ with amplitudes $a = 0, 0.1, 0.5, 1$ and 5. As we see in Figure 1, we have some deviation because of using solutions of ordinary sine-Gordon equation in $t = 0$ as the initial configuration. For this reason the topological charge in the case $a \neq 0$ slightly differs from the case of $a = 0$, and this deviation is increased by increasing a Figure 2 shows the topological charge for a moving kink with initial velocity of $u = 0.3$ in the same potential of the Figure 1

Our numerical calculations show the same results for the anti-kink solution and the two-kink solution with any initial

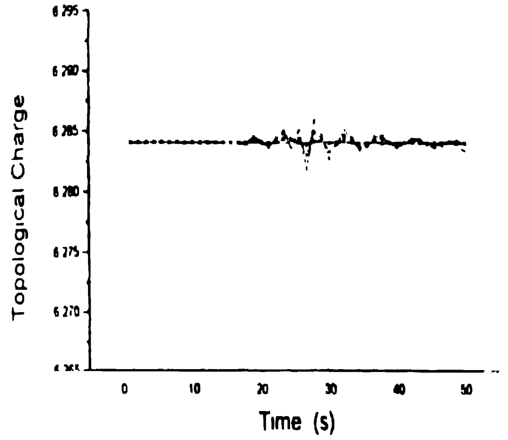

Figure 1. The topological charge of a kink with zero initial velocity in the potential $v(x) = ae^{-5x^2}$ with $a = 0$ (solid), 0.1 (dash) 0.5 (bar), 1 (circle) and 5 (cross).

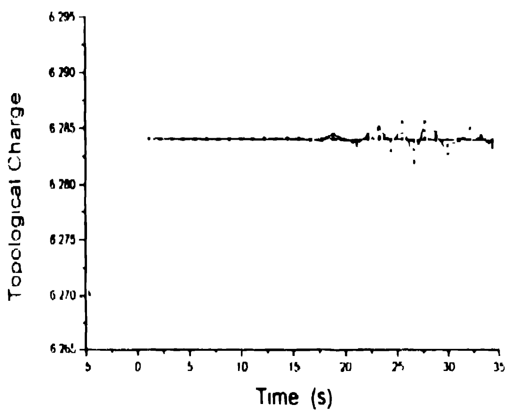

Figure 2. The topological charge of a kink with initial velocity, $u = 0.3$ in the potential $v(x) = ae^{-5x^2}$ with $a = 0$ (solid), 0.1 (dash) 0.5 (bar), 1 (circle) and 5 (cross).

velocity and any slowly varying weak potential (small a). Therefore, we find that the topological charge remains unchanged with a good approximation. In all cases, the deviation due to the presence of potential is less than the numerical errors.

We have also studied the evolution of topological charge for a Delta like potential. As we pointed out at the beginning of this section for simulating a Delta-like function barrier we use $v(x) = ae^{-bx^2}$ with large a and b. This function is narrow and also high, so bahaves as a Delta-like barrier. Figure 3 shows the topological charge in the case of $v(x) = 10e^{-15x^2}$ for a kink with initial velocity of $u = 0.3$.

We find that deviation due to the existence of $v(x)$ is still less than the numerical errors. Numerical simulations show that in the presence of $v(x)$ the energy is also conserved to a very good approximation (see Figure 4).

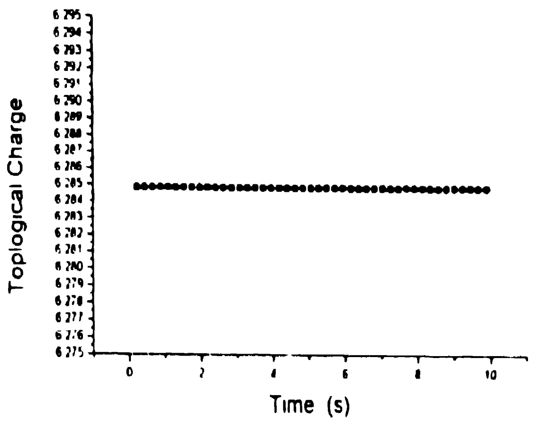

Figure 3. The topological charge of a kink with initial velocity of $u = 0.3$ in the potential $v(x) = 10e^{-15x^2}$

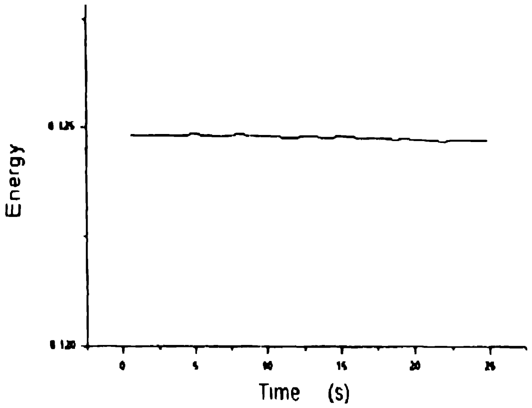

Figure 4. The energy of a kink with zero initial velocity in the potential $v(x) = 0.5e^{-5x^2}$

3.2 Interaction of a soliton with a barrier.

We know that in the case of $v(x) = 0$, a kink with zero initial velocity remains at rest (see Figure 5). Our calculation show

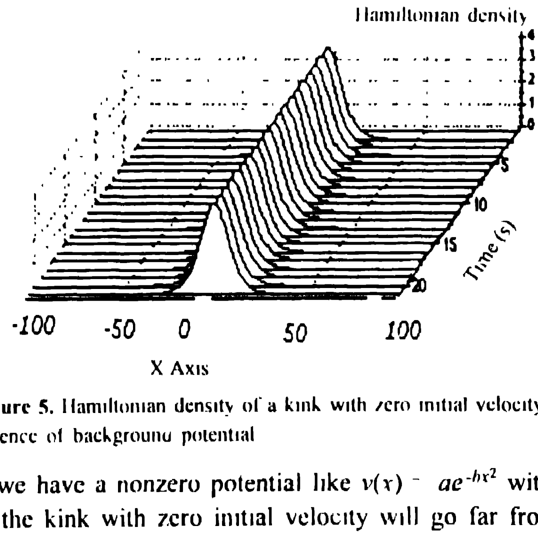

Figure 5. Hamiltonian density of a kink with zero initial velocity in the absence of background potential

if we have a nonzero potential like $v(x) = ae^{-bx^2}$ with $a > 0$, the kink with zero initial velocity will go far from the barrier with an acceleration which depends on the form of $v(x)$. Figure 6 shows the evolution of a kink with zero initial velocity near $v(x)$.

If the kink has a suitable initial velocity depending on the characters of barrier, it will pass through the barrier

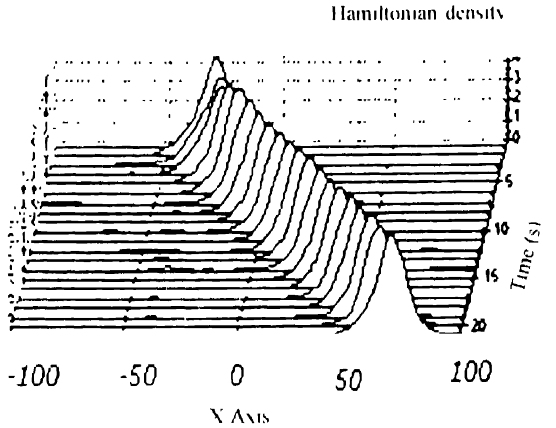

Figure 6. Hamiltonian density of a kink with zero initial velocity in the potential $v(x) = 2e^{-3(x+50)^2}$

(Figure 7). But if its initial velocity (consequently its energy) is not big enough, then it will be reflected from the barrier (Figure 8). But the topological charge and also the energy remain unchanged during the interaction

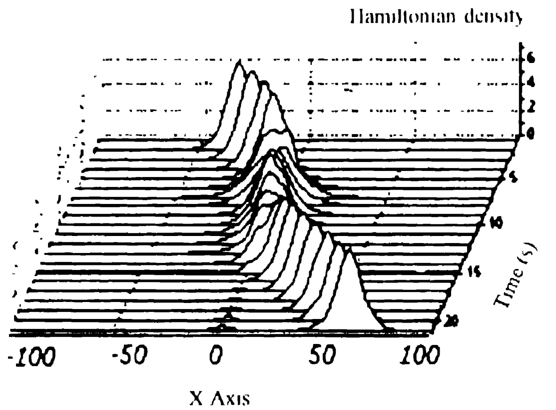

Figure 7. Hamiltonian density of a kink with initial velocity $u = 0.65$ in the potential $v(x) = 2e^{-5x^2}$ It passes through the barrier

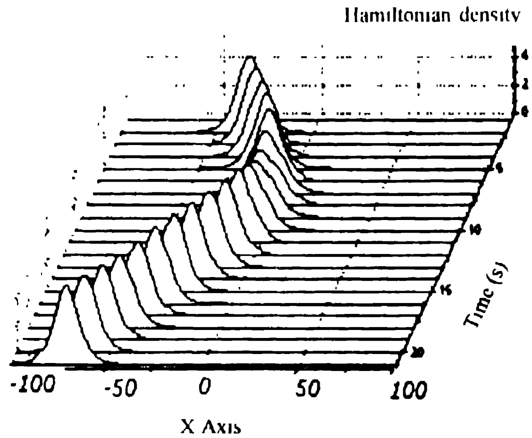

Figure 8. Hamiltonian density of a kink with initial velocity of $u = 0.5$ in the potential $v(x) = 2e^{-5x^2}$ It does not pass

Because of repulsive force of a barrier, an interesting phenomenon occurs for two-kink solution of sine-Gordon equation. In the case of conventional sine-Gordon equation,

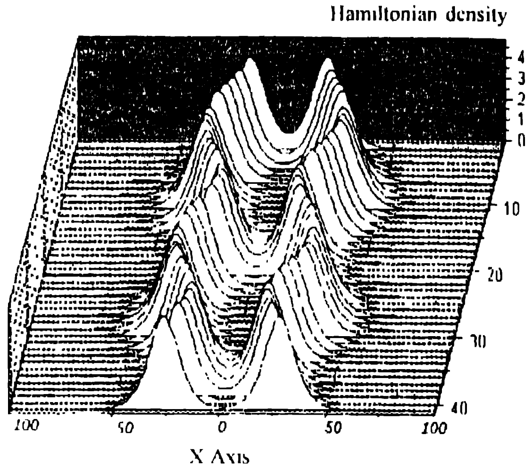

Figure 9. Hamiltonian density for two-kink solution with initial velocity of $u = 0.3$ in potential $v(x) = 0.5(e$ $)$

there is no bound state for the two-kink solution. Now, if we choose $v(x) = a\left(e^{-b(x-c)^2} + e^{-b(x+c)^2}\right)$ and two-kink solution at the origin as the initial configuration, there is a bound state-like system Figure 9 shows a two-kink solution which has bound state in a double barrier potential.

3.3. *Interaction of a soliton with a potential well*

A kink solution is attracted by potential well, like $v(x) = ae^{-bx^2}$ with $a < 0$. If we have a kink with zero initial velocity in a well (outside of the minimum of potential), it will oscilate in the well (Figure 10). If the kink has a suitable velocity, it will leave the well (Figure 11), otherwise it will oscillate in the well.

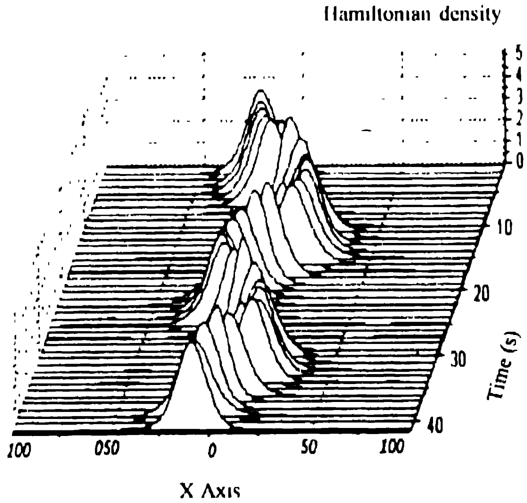

Figure 10. Hamiltonian density of a kink with zero initial velocity in the potential $v(x) = -0.5e^{-5x^2}$

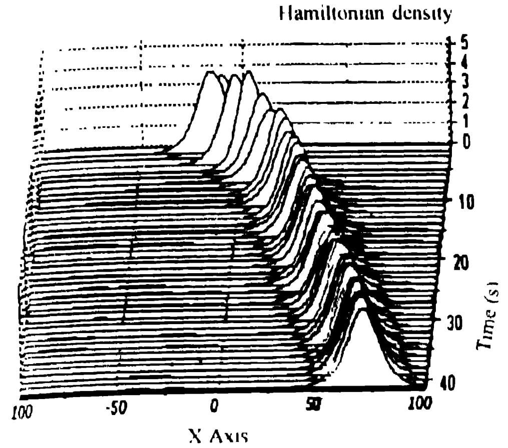

Figure 11. Hamiltonian density of a kink with initial velocity $u = 0.3$, in potential $v(x) = 0.5e^{-x^2}$

3.4 *Interaction of soliton with a Delta function potential*

In sub-Section 3.1, we saw that the topological charge is approximately conserved for potentials like a Delta function, so we may use our method for this potential. In this case as Figures 12 and 13 show, the soliton either is reflected from or transmitted through the barrier depending on its initial velocity (or its energy). Moreover, there are also some radiation which cause the oscillating of the amplitude of the soliton which is commonly observed in the superposition of a soliton with a non-solitonic waves [12]. Figure 14 shows that the topological charge is conserved. Figure 15 shows the total energy as a function of time.

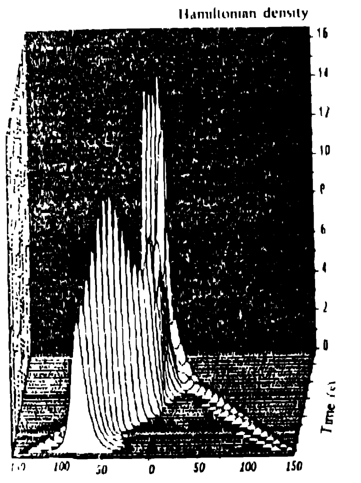

Figure 12. Hamiltonian density of a kink with initial velocity $u = 0.75$ interacts with the potential $v(x) = 7*e^{-10x^2}$

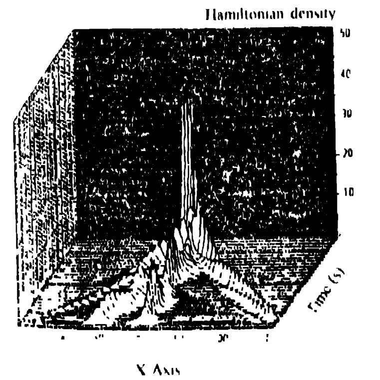

Figure 13. Hamiltonian density of a kink with initial velocity $u = 0.85$ interacts with the potential $v(x) = 7*e^{-10x^2}$

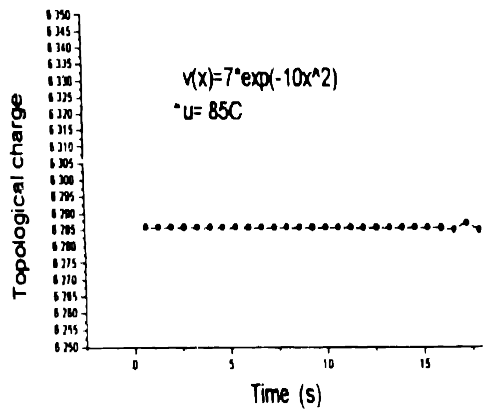

Figure 14. The topological charge for interaction of Figure 13

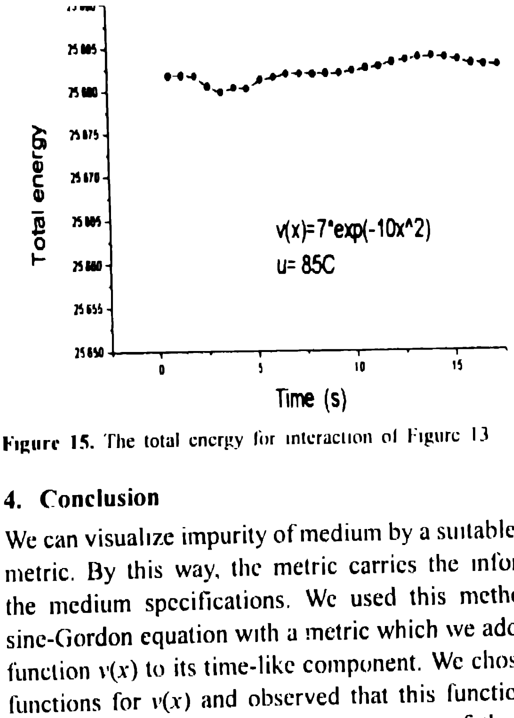

Figure 15. The total energy for interaction of Figure 13

4. Conclusion

We can visualize impurity of medium by a suitable nontrivial metric. By this way, the metric carries the information of the medium specifications. We used this method for the sine-Gordon equation with a metric which we added a small function $v(x)$ to its time-like component. We chose different functions for $v(x)$ and observed that this function play the role of a potential. We see that in most of the cases, the soliton behaves like a particle. Moreover, we observe that the topological charge and the total energy are conserved with a good approximation.

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